Toric fiber products

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Abstract

In this paper, we study the invariance of the toric structure for the fiber products under toric flips. We give some useful criteria and carry it out in the 3-dimensional case.

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1 Introduction

For any two schemes X and Y over a scheme S, the fiber product $X \times_S Y$ exists and is unique up to isomorphisms. The basic applications of fiber products contain the definition of a fiber of a morphism and the notion of base extension, which develop all concepts of algebraic geometry in a relative context. The theory of toric varieties has caught lots of attentions over the past few decades. Besides its many fruitful applications to algebraic geometry, singularity theory and combinatorics, it also provides lots of workable examples in algebraic geometry, due to its highly computability. In particular, it has been used to test some difficult conjectures arising from algebraic geometry — mostly because that it provides a quite different yet elementary way (usually combinatorial way) to see many examples and phenomena in algebraic geometry.

It is natural to ask "Is the fiber product of two toric varieties still toric?" Unfortunately, the answer is "NO". For example, let $\sigma = \langle e_1, e_2\rangle_{\mathbb{R}^+}$ and $\sigma' = \langle e_1, e_1 + e_2\rangle_{\mathbb{R}^+}$ which associate two affine toric varieties $U_\sigma = \operatorname{Spec}\mathbb{C}[x^{e_1^\vee}, x^{e_2^\vee}]$ and $U_{\sigma'} = \operatorname{Spec}\mathbb{C}[x^{e_1^\vee}, x^{e_2^\vee}, x^{e_1^\vee - e_2^\vee}]$. It is easy to see that

$$U_{\sigma'} \times_{U_\sigma} U_{\sigma'} = \operatorname{Spec}\mathbb{C}[u, v, w, w']/\langle u - vw, v(w - w')\rangle$$

and $\langle u - vw, v(w - w')\rangle$ is not a prime ideal, so $U_{\sigma'} \times_{U_\sigma} U_{\sigma'}$ is not a toric variety.

On Birational Geometry, the Minimal Model Program plays an important role, in which the key ingredients consist of flips and flops, so it is a good choice to study toric fiber products under flips and flops first. In the case of a toric flip:

$$\begin{array}{ccc} T_N \subseteq X_\Sigma & \overset{f}{\dashrightarrow} & X_{\Sigma'} \supseteq T_N \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & T_N \subseteq X_{\Sigma_0} & \end{array},$$

we observe that the fiber product $X \coloneqq X_\Sigma \times_{X_{\Sigma_0}} X_{\Sigma'}$ is a toric variety if and only if (1): X is irreducible, (2): X is reduced and (3): the graph closure $\overline{\Gamma}_f$ is a toric variety. Indeed, the identity map of $0^\vee \to 0^\vee$ induces the identity map of $\phi|_{T_N}, \phi'|_{T_N} : T_N \to T_N$, so $T_N \times_{T_N} T_N \subseteq \Gamma_f$. Since X_Σ is irreducible, Γ_f is also irreducible and thus $\overline{T_N \times_{T_N} T_N} = \overline{\Gamma}_f$. If X is irreducible and reduced, then $\overline{\Gamma}_f = X_{\text{red}} = X$ is a toric variety.

The goal of this paper is to study the statements (1), (2), and (3) separately. For (3), I give the equivalent statements only involving dual cones.

Theorem 1.1. Let Σ and Σ' be two different subdivisions of a fan Σ_0. Let $f : X_\Sigma \dashrightarrow X_{\Sigma'}$ be the birational map induced by $X_\Sigma \to X_{\Sigma_0} \leftarrow X_{\Sigma'}$. If we consider the coarsest common subdivision $\widetilde{\Sigma}$ of them, then the following statements are equivalent:

(i) $X_{\widetilde{\Sigma}} \to X_\Sigma \times X_{\Sigma'}$ is a closed immersion,

(ii) $X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f$ is a closed immersion,

(iii) $X_{\widetilde{\Sigma}} \simeq \overline{\Gamma}_f$,

(iv) $\overline{\Gamma}_f$ is a toric variety.

As an application, I give an example whose graph closure is a toric variety (See Theorem 2.2). For (1), I show that X is irreducible for all toric flips.

Theorem 1.2. If we consider the local toric flip

$$\begin{array}{ccc} X_\Sigma & \dashrightarrow & X_{\Sigma'} \\ & \searrow \quad \swarrow & \\ & X_{\Sigma_0} & \end{array}$$

defined by the relation

$$u \coloneqq \sum_{i=0}^{r} a_i v_i = \sum_{i=0}^{s} b_i w_i,$$

then $X \coloneqq X_\Sigma \times_{X_{\Sigma_0}} X_{\Sigma'}$ is irreducible.

As a corollary, I find that the normalization of the fiber product X is $X_{\widetilde{\Sigma}}$, where $\widetilde{\Sigma}$ is the coarsest common subdivision of Σ and Σ' (See Corrollary 3.1). For (2), in the 3-dimensional case, I reduce the problem to the local case (See Theorem 4.1) and give a numerical criterion for them. Let U_{00} be a first affine local chart.

Theorem 1.3. If $g = \gcd(a_0, a_1)$ and $a_i = ga'_i$, then the following statements are equivalent:

(iv) U_{00} is reduced.

(v) For all $0 \le \lambda \le a'_0 a'_1$, there exists a non-negative integer $y \le \lambda / a'_0$ such that

$$\{g\lambda\}_b \ge g \cdot \{\lambda - a'_0 y\}_{a'_1},$$

where $\{n\}_m$ denotes the remainder of n divided by m.

(vi) There exists a non-negative integer $y_0 \le b/a'_0$ such that $a'_1 | (b - a'_0 y_0)$, or equivalently,

$$\{g - 1/a'_0\}_{a'_1} \le b/a'_0.$$

This criterion gives us an easy way to construct examples whose fiber products are toric or not (See Remark 4.1). This paper is organized as follows. In section 2, we study the statement (3); in section 3, we study the statement (1) and in section 4, we study the statement (2).

2 Graph closures

2.1 Warm-up

Let Σ be the fan in N and let Σ' be the subdivision of Σ. The identity map of N induces the toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$. We claim that if $\Sigma' \ne \Sigma$, then

$$X := X_{\Sigma'} \times_{X_\Sigma} X_{\Sigma'}$$

is not irreducible, so it is not a toric variety.

Suppose not, let Z and S be the exceptional locus of $X_{\Sigma'}$ and X_Σ respectively via ϕ. We find that since X is assumed to be irreducible,

$$Z_1 := (Z \times_S Z)_{\text{red}} \hookrightarrow X_{\text{red}} = \overline{T_N \times_{T_N} T_N} = \overline{\Gamma}_{id_{X_{\Sigma'}}} = X_{\Sigma'}$$

can be regarded as a closed subscheme over S, and thus $Z_1 \subseteq Z$. Picking $\tau' \in \Sigma' \setminus \Sigma$ with smallest dimension among $\Sigma' \setminus \Sigma$, if $\tau \in \Sigma$ is the smallest cone containing τ', then $V(\tau') \subset Z$ with the same dimension as Z and $V(\tau) \subset S$. This implies that Z_1 contains an open subset

$$T_{N(\tau')} \times_{T_{N(\tau)}} T_{N(\tau')} \subseteq Z \times_S Z,$$

where $T_{N(\tau)} = \text{Hom}(\tau^\perp \cap M, \mathbb{C}^\times)$. Since $T_{N(\tau')} \to T_{N(\tau)}$ is $T_{N(\tau')}$-equivariant, $g : T_{N(\tau')} \times_{T_{N(\tau)}} T_{N(\tau')} \to T_{N(\tau')}$ has same fiber dimension, and thus

$$\dim T_{N(\tau')} \times_{T_{N(\tau)}} T_{N(\tau')} = \dim T_{N(\tau')} + \dim g^{-1}(z) = \dim Z + \dim g^{-1}(z) > \dim Z$$

for any $z \in T_{N(\tau')}$. But $T_{N(\tau')} \times_{T_{N(\tau)}} T_{N(\tau')} \subseteq Z_1 \subseteq Z$, which leads to a contradiction.

2.2 A criterion of graph closures being toric

Theorem 2.1. Let Σ and Σ' be two different subdivisions of a fan Σ_0. Let $f : X_\Sigma \dashrightarrow X_{\Sigma'}$ be the birational map induced by $X_\Sigma \to X_{\Sigma_0} \leftarrow X_{\Sigma'}$. If we consider the coarsest common subdivision $\widetilde{\Sigma}$ of them, then the following statements are equivalent:

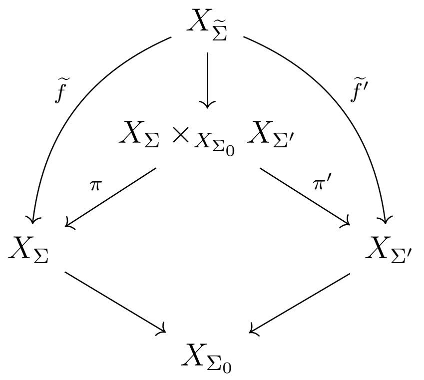
(i) $X_{\widetilde{\Sigma}} \to X_\Sigma \times X_{\Sigma'}$ is a closed immersion,

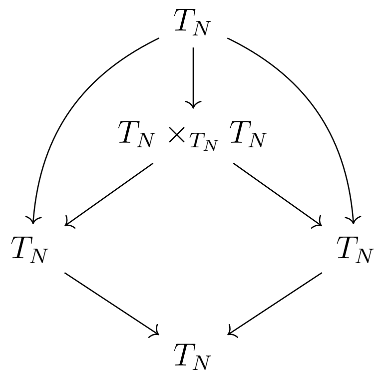
(ii) $X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f$ is a closed immersion,

(iii) $X_{\widetilde{\Sigma}} \simeq \overline{\Gamma}_f$,

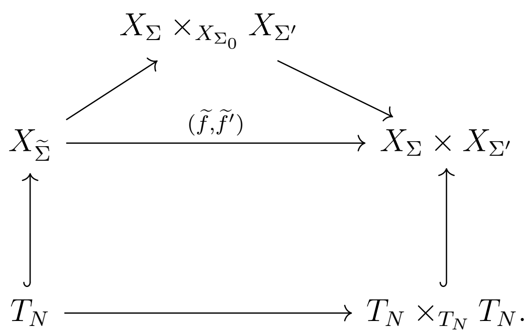
(iv) $\overline{\Gamma}_f$ is a toric variety.

Proof: Consider the following three diagrams:





and



Since $X_{\widetilde{\Sigma}} \to X_\Sigma \times X_{\Sigma'}$ is proper which is closed, we have

$$X_{\widetilde{\Sigma}} = \overline{T_N} \longrightarrow \overline{T_N \times_{T_N} T_N} = \overline{\Gamma}_f.$$

Note that $X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f$ is surjective since it is birational and closed. We can get that

$$X_{\widetilde{\Sigma}} \to X_\Sigma \times X_{\Sigma'} \text{ is closed immersion } \iff X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f \text{ is closed immersion } \iff X_{\widetilde{\Sigma}} \simeq \overline{\Gamma}_f.$$

Moreover, if $\overline{\Gamma}_f$ is a toric variety, then

$$X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f,\ \overline{\Gamma}_f \to X_\Sigma,\ \overline{\Gamma}_f \to X_{\Sigma'}$$

are birational toric morphisms. We may identify $\overline{\Gamma}_f = X_F$ for some fan $F \subset N$. Then F is subdivision of Σ and Σ', and $\widetilde{\Sigma}$ is subdivision of F, so $F = \widetilde{\Sigma}$, i.e. $X_{\widetilde{\Sigma}} \to \overline{\Gamma}_f$ is an isomorphism. Hence

$$X_{\widetilde{\Sigma}} \simeq \overline{\Gamma}_f \iff \overline{\Gamma}_f \text{ is a toric variety.}$$

□

Remark 2.1. It is clear that $X_{\widetilde{\Sigma}} \to X_\Sigma \times X_{\Sigma'}$ is a closed immersion if and only if

$$(\sigma^\vee \cap M) + (\sigma'^\vee \cap M) = (\sigma \cap \sigma')^\vee \cap M \tag{1}$$

for all (maximal cone) σ, σ' contained in some cone of Σ_0.

2.3 The case of flips

Given a relation

$$\sum_{i=0}^{r} a_i v_i = \sum_{j=0}^{s} b_j w_j,$$

where a_i, $b_j \in \mathbb{N}$, v_i, $w_j \in N = \mathbb{Z}^{r+s+1}$ are primitive vectors, and $\{v_0, \dots, v_r, w_0, \dots, w_{s-1}\}$ form a $\mathbb{Q}$ basis of $N \otimes_{\mathbb{Z}} \mathbb{Q}$, the two distinct simplicial subdivisions of $\langle v_0, \dots, v_r, w_0, \dots, w_s\rangle_+$ define a local flip $f : X_\Sigma \dashrightarrow X_{\Sigma'}$ over X_{Σ_0} as in[Rei83], where Σ, Σ', Σ_0 are defined by their top cones as follows:

$$\begin{aligned}
\Sigma(r+s+1) &= \{\sigma_j = \langle v_0, \dots, v_r, w_0, \dots, \widehat{w}_j, \dots, w_s\rangle_+ \mid 0 \le j \le s\}, \\
\Sigma'(r+s+1) &= \{\sigma'_i = \langle v_0, \dots, \widehat{v}_i, \dots, v_r, w_0, \dots, w_s\rangle_+ \mid 0 \le i \le r\}, \\
\Sigma_0(r+s+1) &= \{\tau = \langle v_0, \dots, v_r, w_0, \dots, w_s\rangle_+\}.
\end{aligned}$$

In the special case of $r = s = 1$, we can prove that $\overline{\Gamma}_f$ always is a toric variety.

Theorem 2.2. If Σ and Σ' are two simplicial fans in $N \otimes_{\mathbb{Z}} \mathbb{R}$ defined by

$$u := a_0 v_0 + a_1 v_1 = b_0 w_0 + b_1 w_1,$$

where a_i, $b_j \in \mathbb{N}$, then $\overline{\Gamma}_f$ is a toric variety.

Proof: By Theorem 2.1, it suffices to check (1) holds when $(\sigma, \sigma') = (\sigma_j, \sigma'_i)$ for $i, j \in \{0, 1\}$. In this case, $\sigma \cap \sigma' = \langle \widehat{v}_i, \widehat{w}_j, u\rangle_+$. Let $\{i, k\} = \{0, 1\}$.

- If $j = 1$, then $(\sigma \cap \sigma')^\vee$ is generated by rays

  $$v_i^\vee,\ a_i v_k^\vee - a_k v_i^\vee,\ w_0^\vee.$$

  Hence

  $$\begin{aligned}
  (\sigma \cap \sigma')^\vee \cap M &= \langle v_i^\vee, a_i v_k^\vee - a_k v_i^\vee\rangle_+ \cap M + \langle w_0^\vee\rangle_+ \cap M \\
  &\subseteq (\sigma^\vee \cap M) + (\sigma'^\vee \cap M)
  \end{aligned}$$

- If $j = 0$, by the linear relation $v_k = (u - a_i v_i)/a_k$ and $w_1 = (u - b_0 w_0)/b_1$ we have

  $$\overline{(\sigma \cap \sigma')^\vee \setminus (\sigma^\vee \cup \sigma'^\vee)} = \langle -v_i, -w_0, v_k, w_1, u\rangle_+^\vee = \langle -v_i, -w_0, u\rangle_+^\vee,$$

which is generated by rays

$$v_k^\vee,\ a_i v_k^\vee - a_k v_i^\vee,\ -w_0^\vee.$$

With a similar argument,

$$\overline{(\sigma\cap\sigma')^\vee \setminus (\sigma^\vee \cup \sigma'^\vee)} \cap M \subseteq (\sigma'^\vee \cap M) + (\sigma^\vee \cap M)\,.$$

□

Remark 2.2. Theorem 2.2 may not hold in general for $r \geq 2$. For example, consider two simplicial fans defined by the relation

$$3v_0 + 2v_1 + v_2 = 3w_0 + 2w_1 + w_2,$$

and two cones $\sigma = \sigma_0$, $\sigma' = \sigma_0'$. We find that $\sigma^\vee \cap M + \sigma'^\vee \cap M$ has generating set S consisting of

$$\begin{array}{cccc}
(2,0,0,0,3) & (0,0,1,0,0) & (0,0,2,0,1) & (-1,0,3,0,0) \\
(-1,1,1,0,0) & (0,3,0,2,0) & (-2,3,0,0,0) & (-1,2,0,0,0) \\
(0,1,0,0,0) & (1,0,0,1,0) & (0,1,0,0,1) & (1,0,0,1,0) \\
(0,0,1,-1,2) & (0,0,0,-1,0) & (0,0,0,-1,1). &
\end{array}$$

Note that $(-1,2,0,-1,2) \in (\sigma\cap\sigma')^\vee \cap M$ but not in $\mathbb{Z}_{\geq 0}S$. Hence $\overline{\Gamma}_f$ is not a toric variety, although it is the closure of the torus.

3 Irreducibility of fiber products under toric flips

Theorem 3.1. If we consider the local toric flip

$$\begin{array}{ccc}
X_\Sigma & \dashrightarrow & X_{\Sigma'} \\
& \searrow \quad \swarrow & \\
& X_{\Sigma_0} &
\end{array}$$

defined by the relation

$$u := \sum_{i=0}^{r} a_i v_i = \sum_{i=0}^{s} b_i w_i,$$

then $X := X_\Sigma \times_{X_{\Sigma_0}} X_{\Sigma'}$ is irreducible.

Proof: By symmetry, we only need to prove that $U_\sigma \times_{U_{\sigma_0}} U_{\sigma'}$ is irreducible where $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ are the simplicial cones spanned by $\beta = \{v_0, \ldots, v_r, w_0, \ldots, w_{s-1}\}$, and $\beta' = \{v_1, \ldots, v_r, w_0, \ldots, w_s\}$ respectively and $\sigma_0 = \sigma + \sigma'$. Note that

$$\sigma^\vee \cap (\beta \setminus v_i)^\perp, \sigma'^\vee \cap (\beta' \setminus w_j)^\perp \subseteq \sigma_0^\vee = \sigma^\vee \cap \sigma'^\vee, \quad \forall i \in \{0,1,\ldots,r\},\ j \in \{0,1,\ldots,s\}.$$

Let F be the face of $\sigma^\vee$ generated by $\{\sigma^\vee \cap (\beta \setminus w_i)^\perp\}_{i=0}^{s-1}$ and F' be defined similarly.

Let $\mathcal{X}$ be the set of generators of the cone $\sigma_0^\vee \cap M$ and let $\mathcal{Y}$, $\mathcal{Z} \subset M$ such that $\mathcal{X} \cup \mathcal{Y}$, $\mathcal{X} \cup \mathcal{Z}$ are the sets of generators of the cones $\sigma^\vee \cap M$, $\sigma'^\vee \cap M$ respectively. Let I, J, K be the toric ideals such that

$$\mathbb{C}[\mathcal{X}] = \mathbb{C}[X]/K,\ \mathbb{C}[\mathcal{X} \cup \mathcal{Y}] = \operatorname{Spec} \mathbb{C}[X, Y]/I,\ \mathbb{C}[\mathcal{X} \cup \mathcal{Z}] = \operatorname{Spec} C[X, Z]/J.$$

Clearly, the fiber product $U_\sigma \times_{U_{\sigma_0}} U_{\sigma'}$ is the spectrum of the ring

$$\mathbb{C}[X, Y]/I \otimes_{\mathbb{C}[X]/K} \mathbb{C}[X, Z]/J \simeq \mathbb{C}[X, Y, Z] \big/ I + J.$$

We claim that

$$\mathbb{C}[X, Y, Z] \big/ \sqrt{I + J} \simeq \mathbb{C}[\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}],$$

which is a domain and thus the theorem follows. The claim is equivalent to the statement that $\sqrt{I + J}$ is the "toric ideal" L of $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. Since $I + J \subseteq L$ and L is a prime ideal, $\sqrt{I + J} \subseteq \sqrt{L} = L$. For the converse statement, we discuss it through several steps, see Figure 1.

Step 0. Let $x_i \in \mathbb{Z}_{\geq 0}\mathcal{X}$, $y_i \in \mathbb{Z}_{\geq 0}\mathcal{Y}$ and $z_i \in \mathbb{Z}_{\geq 0}\mathcal{Z}$. Assume that there exists a relation

$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2. \tag{2}$$

For simplicity, let X_i, Y_i, Z_i denote the variables corresponding to the vectors $x_i, y_i, z_i \in M$ respectively. For $i = 1$, 2, if there exists $k_i \in \mathbb{N}$ such that

$$X_i^{k_i}(X_1 Y_1 Z_1 - X_2 Y_2 Z_2) \in \sqrt{I + J},$$

then $X_1 Y_1 Z_1 - X_2 Y_2 Z_2 \in \sqrt{I + J}$, since

$$(X_1 Y_1 Z_1 - X_2 Y_2 Z_2)^{k_1 + k_2 + 1} = \sum_{k=0}^{k_1 + k_2} \binom{k_1 + k_2}{k} (X_1 Y_1 Z_1 - X_2 Y_2 Z_2)(X_1 Y_1 Z_1)^k (X_2 Y_2 Z_2)^{k_1 + k_2 - k}$$

and either $k \geq k_1$ or $k_1 + k_2 - k \geq k_2$ holds. Hence, if necessary, we may replace (2) by two relations

$$(k_1 + 1)x_1 + y_1 + z_1 = (k_1 x_1 + x_2) + y_2 + z_2, \tag{2.1}$$

$$(k_2 x_2 + x_1) + y_1 + z_1 = (k_2 + 1)x_2 + y_2 + z_2. \tag{2.2}$$

Of course, the pair (x_1, x_2) can be replaced by any pair in $\{x_1, y_1, z_1\} \times \{x_2, y_2, z_2\}$. This process of replacements will occur frequently in the following algorithm of finding such pair (k_1, k_2).

Step 1. If $x_1, x_2 \in \sigma_0^\vee \setminus (\partial\sigma^\vee \cap \sigma'^\vee)$, then we take k_1, $k_2 \in \mathbb{N}$ such that $x_i + \frac{1}{k_i} z_j \in \sigma_0^\vee$ for all $j \in \{1, 2\}$, i.e., $x_{ij} := k_i x_i + z_j \in \mathbb{Z}_{\geq 0}\mathcal{X}$. The relation (2.1) can be decomposed as

$$\begin{aligned} k_1 x_1 + z_1 &= x_{11} \\ x_{12} &= k_1 x_1 + z_2 \\ x_{11} + x_1 + y_1 &= x_{12} + x_2 + y_2. \end{aligned}$$

The first two relations hold in $\mathcal{X} \cup \mathcal{Z}$, and the last one is the relation in $\mathcal{X} \cup \mathcal{Y}$. Hence, the binomial corresponding to the relation (2.1) is in $I + J$. A similar argument holds for (2.2).

Step 2. We consider either $x_1 \in (\partial\sigma^\vee \cap \sigma'^\vee) \setminus \{0\}$ or $x_2 \in (\partial\sigma^\vee \cap \sigma'^\vee) \setminus \{0\}$. For such $x_i \in (\partial\sigma^\vee \cap \sigma'^\vee) \setminus \{0\}$, we take $k_i \in \mathbb{N}$ such that $k_i x_i = x_i' + z_i'$ where $x_i' \in \sigma_0^\vee \setminus (\partial\sigma^\vee \cap \sigma'^\vee)$ and $z_i' \in F'$. Without loss of generality, we assume that $x_1 \in (\partial\sigma^\vee \cap \sigma'^\vee) \setminus \{0\}$.

If $x_2 \neq 0$, then either $x_2 \in (\partial\sigma^\vee \cap \sigma'^\vee) \setminus \{0\}$ or $x_2 \in \sigma_0^\vee \setminus (\partial\sigma^\vee \cap \sigma'^\vee)$. For the latter one, we take k_2 as in *Step 1* and then replace (2) with (2.1)+(2.2) by *Step 0*, so that we can reduce it to *Step 1*.

If $y_2 \notin F$, then we take $k_2 \in \mathbb{N}$ such that $k_2 y_2 = x_2' + y_2'$ where $y_2' \in F$ and $x_2' \in \sigma_0^\vee \setminus (\partial\sigma'^\vee \cap \sigma^\vee)$ and replace (2) by (2.1) and (2.2) on the pairs (x_1, y_2) and (k_1, nk_2) for $n \in \mathbb{N}$ as in *Step 0*. Now we can reduce it to *Step 1* for (2.1) and produce the corresponding equations (2.1.1) and (2.1.2). For (2.2), we can rewrite it (modulo I) as

$$x_1 + (n(y_2' + x_2') + y_1) + z_1 = x_2 + n(y_2' + x_2') + y_2 + z_2.$$

and replace it by

$$x_1 + (y_1 + nx_2') + z_1 = x_2 + (y_2 + nx_2') + z_2. \tag{3}$$

When we take $n \gg 0$ such that $y_i + nx_2' \in \mathbb{Z}_{\geq 0}\mathcal{X}$ (for $i = 1, 2$), it is similarly reduced to *Step 1*. By symmetry, the similar argument can be applied for $z_2 \notin F'$.

In this step, the remaining case is when $x_2 = 0$, $y_2 \in F$, $z_2 \in F'$, but this is vacuous. Otherwise, $F + F' = u^\perp \cap (\sigma \cap \sigma')^\vee$ is the face of $(\sigma \cap \sigma')^\vee$, which implies $x_1 \in \mathbb{Z}_{\geq 0}\mathcal{X} \cap (F + F') = \{0\}$ ($\rightarrow\!\leftarrow$).

Step 3. The remaining job for us is when either $x_1 = 0$ or $x_2 = 0$. Without loss of generality, we assume that $x_2 = 0$.

If $y_2 \notin F$ or $z_2 \notin F'$, then we have $x_1 \neq 0$ or $y_1 \notin F$ or $z_1 \notin F'$, since $F + F'$ is the face of $(\sigma \cap \sigma')^\vee$. Use the same decomposition of y_i or z_i as in *Step 2* and reduce it to the case of x_1 and x_2 being nonzero.

If $y_2 \in F$ and $z_2 \in F'$, then we have $x_1 = 0$, $y_1 \in F$ and $z_1 \in F'$, since $F + F'$ is the face of $(\sigma \cap \sigma')^\vee$. Let $\sigma^\vee \cap (\beta \setminus w_j)^\perp$, $\sigma'^\vee \cap (\beta' \setminus v_i)^\perp$ be generated by $\widetilde{y}_j, \widetilde{z}_i \in M$ respectively. Note that $\{\widetilde{y}_j, \widetilde{z}_i \mid 0 \leq j \leq s-1, 1 \leq i \leq r\}$ is a linearly independent set over $\mathbb{R}$, otherwise, there exist $p_j, q_i \in \mathbb{R}$ such that

$$\sum_{j=0}^{s-1} p_j \widetilde{y}_j + \sum_{i=1}^{r} q_i \widetilde{z}_i = 0,$$

and then pairing with w_j and v_i, we get $p_j = q_i = 0$. Hence, $y_1 + z_1 = y_2 + z_2$ implies $y_1 = y_2$ and $z_1 = z_2$, and thus

$$Y_1 Z_1 - Y_2 Z_2 = 0 \in I + J.$$

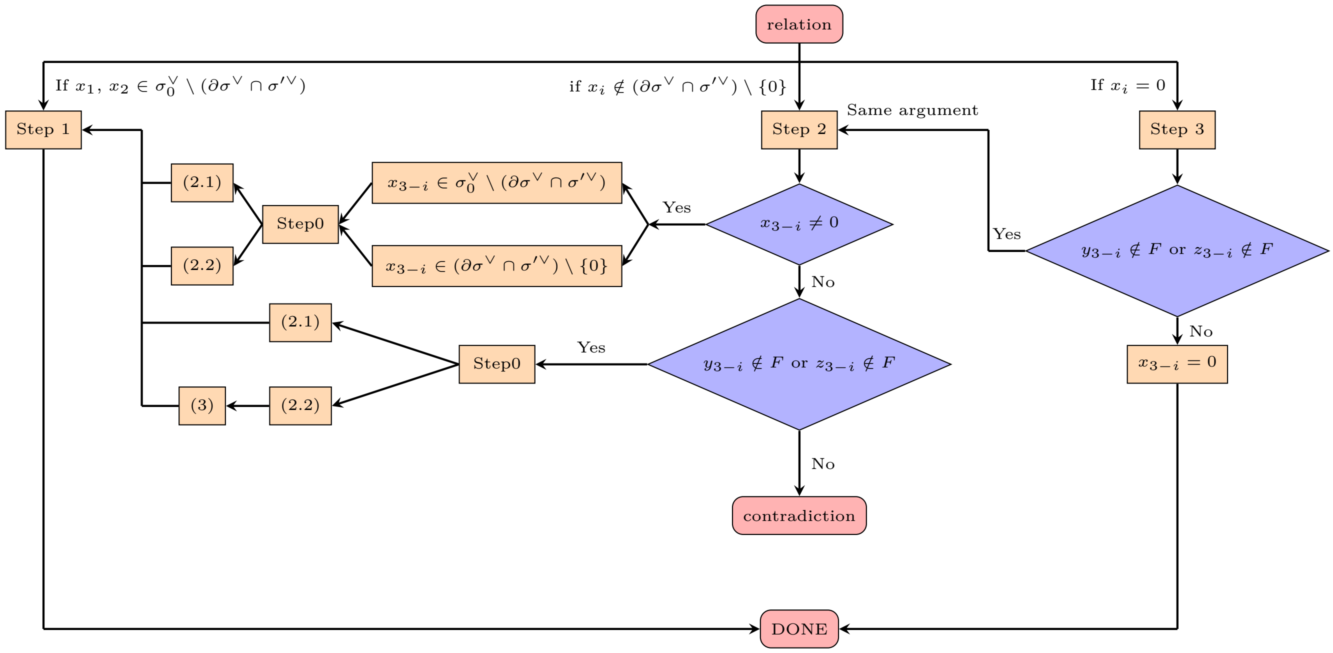


Figure 1: The flow chart for the proof.

□

Remark 3.1. The similar proof works on the local toric flips with general exceptional locus $S \subseteq X_{\Sigma_0}$ as in[Rei83].

Corollary 3.1. Under the assumption in theorem 3.1, the normalization of the fiber product X is $X_{\widetilde{\Sigma}}$, where $\widetilde{\Sigma}$ is the coarsest common subdivision of Σ and Σ'. Moreover, if Σ and Σ' satisfy the condition (1), then X_{red} is already normal and $X_{\mathrm{red}} = X_{\widetilde{\Sigma}}$.

Proof: Recall that the integral closure of the semi-group ring $\mathbb{C}[S]$ in $\mathbb{C}[M]$ is $\mathbb{C}[S^{\mathrm{sat}}]$, where S^{sat} is the saturation of S in the lattice M. Since $u = \sum a_i v_i = \sum b_i w_i$,

$$\mathbb{R}_{\geq 0}(\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}) = \sigma^\vee + \sigma'^\vee = (\sigma \cap \sigma')^\vee,$$

and thus

$$\mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})^{\mathrm{sat}} = (\sigma \cap \sigma')^\vee \cap M.$$

Moreover, the condition (1) implies that $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$ generate $(\sigma \cap \sigma')^\vee \cap M$, and thus $X_{\mathrm{red}} = X_{\widetilde{\Sigma}}$ is normal. □

4 The property about being reduced

In this section, we focus on the case of $r = s = 1$. Consider

$$u := a_0 v_0 + a_1 v_1 = b w_0 + w_1,$$

where $\{v_0, v_1, w_1\}$ forms a basis of $N = \mathbb{Z}^3$ and $b = a_0 + a_1 - 1$. For the sake of convenience, let $U_{ij} := U_{\sigma_i} \times_{U_\sigma} U_{\sigma'_j}$. By Theorem 2.1, Theorem 2.2 and 3.1, we get that the following statements are equivalent:

(i) $X = X_\Sigma \times_{X_{\Sigma_0}} X_{\Sigma'}$ is toric variety,

(ii) $X = X_{\widetilde{\Sigma}}$,

(iii) X is reduced.

We intend to study when X is reduced. Since "being reduced" is a local property, we may work on one affine piece.

Lemma 4.1. If $g = \gcd(a_0, a_1)$ and $a_i = ga_i'$, then the following statements are equivalent:

(iv) U_{00} is reduced.

(v) For all $0 \le \lambda \le a_0'a_1'$, there exists a non-negative integer $y \le \lambda/a_0'$ such that

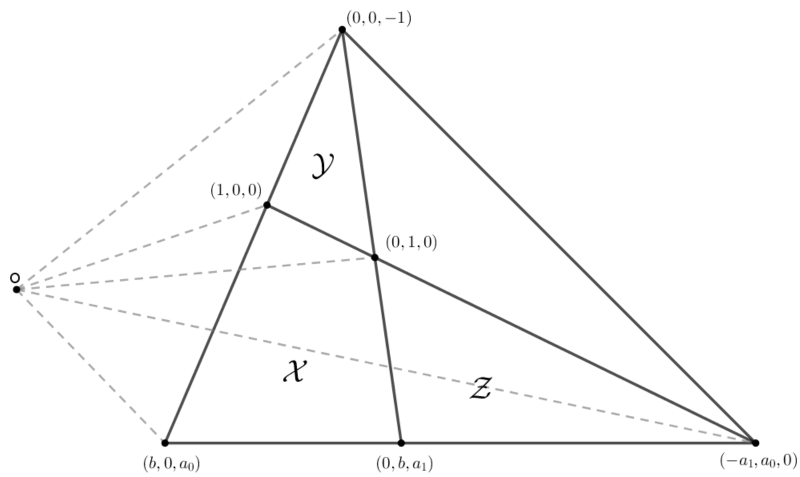
$$\{g\lambda\}_b \ge g \cdot \{\lambda - a_0'y\}_{a_1'},$$

where $\{n\}_m$ denotes the remainder of n divided by m.

(vi) There exists a non-negative integer $y_0 \le b/a_0'$ such that $a_1'|(b - a_0'y_0)$, or equivalently,

$$\{g - 1/a_0'\}_{a_1'} \le b/a_0'.$$

Proof: For simplicity, we assume that $v_0, v_1, w_0 = e_1, e_2, e_3$ and calculate the dual cone, as illustrated in the following figure.



Note that the coordinates in the figure only represent points on that ray, not that these six points lie in the same plane. Now, we use the same notations in the proof of Theorem 3.1. We can get that U_{00} is reduced if and only if $X_1Y_1Z_1 - X_2Y_2Z_2 \in I + J$ with $x_1+y_1+z_1 = x_2+y_2+z_2$. The key point is that the generating set $\mathcal{Y}$ can be selected to be $\{y = (0, 0, -1)\}$.

For any $z \in \Gamma := M \cap (\langle (0,1,0), (-a_1, a_0, 0), (0, b, a_1)\rangle_+ \setminus \langle (0,1,0), (-a_1, a_0, 0)\rangle_+)$, it is clear that $z' = y + z \in \mathbb{Z}_{\ge 0}(\mathcal{X} \cup \mathcal{Z})$. If U_{00} is reduced, then $YZ - Z' \in I + J$. Note that an element in $I + J$ containing the term Y is of the form

$$\sum_i c_i (Y^{n_i} X_{1,i} - X_{2,i}) Y^{m_i} Z_i + \sum_j d_j Y^{n_j} (X_{3,j} Z_{3,j} - X_{4,j} Z_{4,j}),$$

where $n_i y + x_{1,i} = x_{2,i}$, $x_{3,j} + z_{3,j} = x_{4,j} + y_{4,j}$, and c_i, $d_j \in \mathbb{C}$. Since $YZ - Z'$ is not divided by Y, we conclude that $z \in (\mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0) + \mathbb{Z}_{\geq 0}\mathcal{Z}$. Moreover, let $z_0 = z \in \Gamma$ and write $z_0 = x_1 + z_1$ where $x_1 \in \mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0$ and $z_1 \in \mathbb{Z}_{\geq 0}\mathcal{Z}$. If $z_1 \in \Gamma$, then we can again write $z_1 = x_2 + z_2$ where $x_2 \in \mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0$ and $z_2 \in \mathbb{Z}_{\geq 0}\mathcal{Z}$. We keep doing this process until z_{i+1} does not lie in Γ. We claim that this process will terminate. Otherwise, since the 2nd and 3rd coordinates of a lattice point in $\mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z})$ are non-negative integers, the 2nd and 3rd coordinates of $\{z_i\}_{i\geq 0}$ form two sequences of decreasing non-negative integers and thus will be stable. Similarly, the 1st coordinates of $\{z_i\}_{i\geq 0}$ forms a sequence of decreasing integer. Pick $n \in \mathbb{N}$ such that $\{z_i\}_{i\geq n}$ has the same 2nd and 3rd coordinates, which implies that $z_n + \mathbb{Z}e_1^\vee \subseteq \Gamma$ and contradicts the fact that $\{z_i\}_{i\geq 0}$ is an infinite sequence. Hence we conclude that

$$\Gamma \subseteq (\mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0) + \langle (0,1,0), (-a_1, a_0, 0) \rangle \cap M \tag{4}$$

if U_{00} is reduced.

Conversely, we claim that if (4) holds, then U_{00} is reduced. We define

$$\Gamma' = \langle (0,1,0), (-a_1, a_0, 0) \rangle \cap M.$$

For $z \in \Gamma$, say $z = x + z''$ where $x \in \mathbb{Z}_{\geq 0}\mathcal{X}$ and $z'' \in \Gamma'$, let $z' = y + z \in \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z})$ and then $z' = (x + y) + z''$. If $(x + y) \notin \mathbb{Z}_{\geq 0}\mathcal{X}$, then

$$\begin{aligned} x + y + z'' \in& (\langle (0,0,-1), (1,0,0), (0,1,0) \rangle_+ \setminus \langle (1,0,0), (0,1,0)) \rangle_+) + \Gamma' \\ \subseteq& \langle (0,0,-1), (1,0,0), (-a_1, a_0, 0) \rangle_+ \setminus \langle (1,0,0), (-a_1, a_0, 0) \rangle_+, \end{aligned}$$

which contradicts to $z' \in \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z})$. Hence $x' := x + y \in \mathbb{Z}_{\geq 0}\mathcal{X}$, and thus

$$X' - XY \in I, XZ'' - Z, X'Z'' - Z' \in J \implies YZ - Z' \in I + J.$$

Suppose that there exists a relation

$$x_1 + n_1 y + z_1 = x_2 + n_2 y + z_2$$

such that $X_1 Y^{n_1} Z_1 - X_2 Y^{n_2} Z_2 \notin I + J$ for some n_1, $n_2 \geq 0$. We may assume that $n_2 = 0$ by eliminating $\min\{n_1, n_2\}y$, and $n_1 \geq 0$ is the smallest integer such that the above relation holds (note that $n_1 \neq 0$). We find that $z_1 \notin \Gamma$, otherwise, we can replace the original relation with $x_1 + (n_1 - 1)y + z_1' = x_2 + z_2$, where $z_1' = y + z_1$. By the same proof as above, we have $x_1 + n_1 y_1 \in \mathbb{Z}_{\geq 0}\mathcal{X}$, and thus $X_1 Y^{n_1} Z_1 - X_2 Z_2 \in I + J$ ($\rightarrow\!\leftarrow$).

After determining the equivalent statement for (iv), it suffices to show that $\Gamma \subseteq (\mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0) + \Gamma'$ is equivalent to the condition (v). Note that $\Gamma \subseteq (\mathbb{Z}_{\geq 0}\mathcal{X} \setminus 0) + \Gamma'$ is equivalent to $\overline{\Gamma} \subseteq \mathbb{Z}_{\geq 0}\mathcal{X} + \Gamma'$. Given $z = (-p_0, p_1, q) \in \overline{\Gamma}$, that is, $p_0 \geq 0$ $q \geq 0$ and $a_1 p_1 \geq a_0 p_0 + bq$, we want to find a lattice point $z'' = (-\alpha, \beta, 0) \in \Gamma'$ such that $z - z'' = (\alpha - p_0, p_1 - \beta, q) \in \mathbb{Z}_{\geq 0}\mathcal{X}$, that is, there exist α, $\beta \geq 0$ such that

$$a_1\beta \geq a_0\alpha, \alpha \geq p_0, \beta \leq p_1, k := -a_0 p_0 + a_1 p_1 - bq \geq a_1\beta - a_0\alpha. \tag{5}$$

Note that if (α, β) satisfies (5), then $(\alpha, \lceil \frac{a_0\alpha}{a_1} \rceil)$ satisfies (5). Thus, (5) is equivalent to the existence of an integer $\alpha \in [p_0, a_1p_1/a_0]$ such that

$$k \geq a_1 \left\lceil \frac{a_0\alpha}{a_1} \right\rceil - a_0\alpha = g \cdot \{-a_0'\alpha\}_{a_1'}.$$

Hence, $\overline{\Gamma} \subseteq \mathbb{Z}_{\geq 0}\mathcal{X} + \Gamma'$ is also equivalent to the statement that for all $p_0, q_0 > 0$ such that $a_1'p_1 \geq a_0'p_0$, we have

$$P(p_0, q_0) : \exists p_0 \leq \alpha \leq \frac{a_1p_1}{a_0} \text{ such that } \{g(a_1'p_1 - a_0'p_0)\}_b \geq g \cdot \{-a_0'\alpha\}_{a_1'}.$$

Note that if $a_1p_1/a_0 - p_0 \geq a_1'$, then there exists an integer $\alpha \in [p_0, a_1p_1/a_0]$ such that $a_1'|\alpha$, ensuring that the above inequality holds. Therefore, we only need to check it for $\lambda := a_1'p_1 - a_0'p_0 \leq a_0'a_1'$. We take $x_0, x_1 \in \mathbb{N}$ such that $a_0'x_0 - a_1'x_1 = 1$. Since $P(p_0, q_0)$ holds if and only if $P(p_0 + a_1', p_1 + a_0')$ holds, we only need to check that $(p_0, q_0) = (\lambda x_0, \lambda x_1)$ for $0 \leq \lambda \leq a_0'a_1'$. Let $\alpha = \lambda x_0 + y$. Clearly, $P(\lambda x_0, \lambda x_1)$ can be reformulated as

$$\exists\ 0 \leq y \leq \frac{a_1' \cdot \lambda x_1}{a_0} - \lambda x_0 = \frac{\lambda}{a_0} \text{ such that } \{g\lambda\}_b \geq g \cdot \{-a_0'(\lambda x_0 + y)\}_{a_1'} = g \cdot \{\lambda - a_0'y\}_{a_1'}.$$

This proves that (iv) $\iff$ (v).

For (v) $\implies$ (vi), simply by taking $\lambda = b$.

For (vi) $\implies$ (v), let $\lambda = pb + q(a_0' + a_1') + r$ where

$$0 \leq q \leq g - 1 \text{ and } 0 \leq r \leq a_0' + a_1' - 1 - \delta_{q,g-1}.$$

We take $y = py_0 + q \leq pb/a_0' + q \leq \lambda/a_0'$ and get that

$$\{g\lambda\}_b = \{gr + q\}_b = gr + q,$$

since $gr + q \leq g(a_0' + a_1' - 1) + g - 1 = b$ and the equality will not hold. On the other hand,

$$\{\lambda - a_0'y\}_{a_1'} = \{p(b - a_0'y_0) + qa_0' + r - qa_0'\}_{a_1'} = \{r\}_{a_1'}.$$

Hence

$$\{g\lambda\}_b = gr + b \geq g\{r\}_{a_1'} = \{\lambda - a_0'y\}_{a_1'}.$$

□

Remark 4.1. Note that the condition (vi) is equivalent to that

$$\exists y_0, y_1 \in \mathbb{Z}_{\geq 0} \text{ such that } b = a_0'y_0 + a_1'y_1, \tag{vi'}$$

so the condition (vi) is symmetric in a_0' and a_1', that is, U_{00} is reduced if and only if U_{01} is reduced.

By the elementary number theory, if $b \geq (a_0' - 1)(a_1' - 1)$, then the condition (vi') will hold. According to the theorem below, in this case, the fiber product X is a toric variety.

It is clear that if $(a_0', a_0' + a_1')$ satisfies (vi'), then (a_0', a_1') also satisfies (vi'). In other word, if (a_0', a_1') does not satisfy (vi'), then $(a_0', a_0' + a_1')$ also doesn't satisfy (vi'). Hence we can construct the fiber product X, which is not a toric variety in this case; for example, taking $(a_0, a_1) = (3, 5 + 3n)$ for $n \in \mathbb{N}$.

The main result of this section is the following theorem.

Theorem 4.1. X is reduced if and only if U_{00} is reduced.

Before giving the proof, we have to show the following lemma.

Lemma 4.2. If $\pi : X \to X_\Sigma$ and $\mathscr{I}$ is the nilradical ideal of $\mathcal{O}_X$, then

$$\pi_*\mathscr{I} = R^i\pi_*\mathscr{I} = R^i\pi_*\mathcal{O}_X = 0$$

for all $i > 0$, and $\pi_*\mathcal{O}_X = \mathcal{O}_{X_\Sigma}$. Similar statement holds for $\pi' : X \to X_{\Sigma'}$.

Proof: Since π is a proper morphism with fiber dimension ≤ 1, by the formal function theorem, $R^i\pi_*\mathscr{I} = R^i\pi_*\mathcal{O}_X = 0$ for all $i > 1$. For $i = 1$, it suffices to show that

$$H^1(\pi^{-1}(U_{\sigma_j}), \mathscr{I}) = H^1(\pi^{-1}(U_{\sigma_j}), \mathcal{O}_X) = 0, \text{ for } j = 0, 1.$$

Since $\pi^{-1}(U_{\sigma_j})$ is covered by $\left\{U_{jk} \coloneqq U_{\sigma_j} \times_{U_\sigma} U_{\sigma'_k}\right\}_{k=0,1}$, by Čech cohomology, it suffices to show that

$$H^0(U_{j0}, \mathscr{I}) \oplus H^0(U_{j1}, \mathscr{I}) \longrightarrow H^0(U_{j0} \cap U_{j1}, \mathscr{I}) = H^0(U_{\sigma_j} \times_{U_\sigma} U_{\sigma'_0 \cap \sigma'_1}, \mathscr{I}) \tag{6}$$

$$H^0(U_{j0}, \mathcal{O}_X) \oplus H^0(U_{j1}, \mathcal{O}_X) \longrightarrow H^0(U_{j0} \cap U_{j1}, \mathcal{O}_X) = H^0(U_{\sigma_j} \times_{U_\sigma} U_{\sigma'_0 \cap \sigma'_1}, \mathcal{O}_X) \tag{7}$$

are surjective.

By the relation $a_0v_0 + a_1v_1 = bw_0 + w_1$, we have $(\sigma'_0 \cap \sigma'_1)^\vee = \sigma_0'^\vee \cup \sigma_1'^\vee$, and thus

$$0 \to \mathbb{C}[\sigma_0'^\vee \cap \sigma_1'^\vee \cap M] \to \mathbb{C}[\sigma_0'^\vee \cap M] \oplus \mathbb{C}[\sigma_1'^\vee \cap M] \to \mathbb{C}[(\sigma'_0 \cap \sigma'_1)^\vee \cap M] \to 0. \tag{8}$$

Since (7) is equal to the last morphism in (8) tensoring $\mathbb{C}[\sigma_j^\vee \cap M]$ over $\mathbb{C}[\sigma^\vee \cap M]$, we conclude that (7) is surjective.

Using the same notation in the proof of Theorem 3.1, let

$$\sigma^\vee \cap M = \mathbb{Z}_{\geq 0}\mathcal{X}, \ \sigma_j^\vee \cap M = \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Y}), \ \sigma_k'^\vee \cap M = \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z}_k),$$

and let I, J_k, K be the toric ideals of $\mathcal{X} \cup \mathcal{Y}$, $\mathcal{X} \cup \mathcal{Z}_k$, $\mathcal{X} \cup \mathcal{Z}_0 \cup \mathcal{Z}_1$ respectively. According to the proof of Theorem 3.1, $\sqrt{I + K}$ is the "toric ideal" of $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}_0 \cup \mathcal{Z}_1$. Hence $\sqrt{I + K}/(I + K)$ is generated by $\overline{Y_0Z_0 - Y_1Z_1}$, where $y_i \in \mathbb{Z}_{\geq 0}\mathcal{Y}$, $z_i \in \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z}_0 \cup \mathcal{Z}_1)$.

If $z_0, z_1 \in \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z}_k)$ for some $k \in \{0, 1\}$, then $\overline{Y_0Z_0 - Y_1Z_1} \in \sqrt{I + J_k}/(I + J_k)$ maps to $\overline{Y_0Z_0 - Y_1Z_1} \in \sqrt{I + K}/(I + K)$.

If not, we can assume that $z_i \in \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Z}_i)$ and then

$$y_0 + z_0 = y_1 + z_1 \in (\sigma_j^\vee + \sigma_0'^\vee) \cap (\sigma_j^\vee + \sigma_1'^\vee) \cap M = \sigma_j^\vee \cap M = \mathbb{Z}_{\geq 0}(\mathcal{X} \cup \mathcal{Y}),$$

say $y_0 + z_0 = y_1 + z_1 = y_2 + x_2$, where $y_2 \in \mathbb{Z}_{\geq 0}\mathcal{Y}$ and $x_2 \in \mathcal{Z}_{\geq 0}\mathcal{X}$. We have that

$$(\overline{Y_0Z_0 - Y_2X_2}, \overline{Y_1Z_1 - Y_2X_2}) \mapsto \overline{Y_0Z_0 - Y_1Z_1},$$

so (6) is surjecitve.

Note that

$$0 \to \pi_*\mathscr{I} \to \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X_{\mathrm{red}}} \to R^1\pi_*\mathscr{I} = 0. \tag{9}$$

By Corollary 3.1 and the proof of Theorem 2.2, $X_{\mathrm{red}} = X_{\widetilde{\Sigma}}$ is integral. By the proof of Zariski main theorem in[Har77], $\pi_*\mathcal{O}_{X_{\mathrm{red}}} = \mathcal{O}_{X_\Sigma}$. By functoriality, the morphism between structure sheaves $\mathcal{O}_{X_\Sigma} \to \pi_*\mathcal{O}_X$ gives a lifting of (9), and thus the short exact sequence (9) splits.

Now the remaining part is to prove that $\pi_*\mathscr{I} = 0$. By definition and $R^1\pi_*\mathcal{O}_X = 0$, we have

$$0 \to \mathcal{O}_X(\pi^{-1}U_{\sigma_i}) \to \mathcal{O}_X(U_{i0}) \oplus \mathcal{O}_X(U_{i1}) \to \mathcal{O}_X(U_{\sigma_i} \times_{U_\sigma} U_{\sigma_0'\cap\sigma_1'}) \to 0,$$

or rewrite it as

$$\begin{aligned} 0 &\to \mathbb{C}[\sigma_i^\vee \cap M] \oplus \Gamma(\pi_*\mathscr{I}, U_{\sigma_i}) \\ &\to \mathbb{C}[\sigma_i^\vee \cap M] \otimes_{\mathbb{C}[\sigma^\vee\cap M]} \left(\mathbb{C}[\sigma_0'^\vee \cap M] \oplus \mathbb{C}[\sigma_1'^\vee \cap M]\right) \\ &\xrightarrow{\alpha} \mathbb{C}[\sigma_i^\vee \cap M] \otimes_{\mathbb{C}[\sigma^\vee\cap M]} \mathbb{C}[(\sigma_0' \cap \sigma_1')^\vee \cap M] \to 0. \end{aligned} \tag{10}$$

From the long exact sequence induced by (8) $\otimes_{\mathbb{C}[\sigma^\vee\cap M]} \mathbb{C}[\sigma_i^\vee \cap M]$, the kernel of α is

$$\mathbb{C}[\sigma_i^\vee \cap M] \Big/ \mathrm{Im}\left(\mathrm{Tor}_1^{\mathbb{C}[\sigma^\vee\cap M]}\left(\mathbb{C}[(\sigma_0' \cap \sigma_1')^\vee \cap M], \mathbb{C}[\sigma_i^\vee \cap M]\right) \to \mathbb{C}[\sigma_i^\vee \cap M]\right).$$

So we can conclude that $\Gamma(\pi_*\mathscr{I}, U_{\sigma_i}) = 0$. Hence $\pi_*\mathscr{I}|_{U_{\sigma_i}} = 0$ for $i = 0, 1$, that is, $\pi_*\mathscr{I} = 0$. □

Proof: (of Theorem 4.1) Suppose that U_{00} is reduced. By Remark 4.1, U_{10} is also reduced, and thus

$$\mathrm{Supp}\,\mathscr{I} \subseteq (Z \times_S Z')_{\mathrm{red}} \setminus (U_{00} \cup U_{10}) = (\pi')^{-1}(p),$$

where p is the unique point in $V(\sigma_0' \cap \sigma_1') \setminus U_{\sigma_0'} = V(\sigma_1')$. Note that $(Z \times_S Z')_{\mathrm{red}} = \mathbb{P}^1 \times \mathbb{P}^1$, and $\pi|_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\pi'|_{\mathbb{P}^1\times\mathbb{P}^1}$ are projection onto each component. We take a section $s : Z = \mathbb{P}^1 \xrightarrow{\sim} (\pi')^{-1}(p)_{\mathrm{red}} = \mathbb{P}^1$ such that $\pi \circ s = \mathrm{id}_Z$. Let $\mathscr{F} = (s^{-1})_*\iota'^{-1}\mathscr{I}$ be the sheaf on Z, where $\iota' : (\pi')^{-1}(p)_{\mathrm{red}} \hookrightarrow X$. Since $\mathscr{I}$ has the support on $(\pi')^{-1}(p)_{\mathrm{red}}$ and by Lemma 4.2, we have

$$\iota_*\mathscr{F} = \pi_*\iota'_*s_*(s^{-1})_*\iota'^{-1}\mathscr{I} = \pi_*\mathscr{I} = 0,$$

where $\iota : Z \hookrightarrow X_\Sigma$. This implies $\mathscr{F} = 0$, and thus $\mathscr{I} = 0$, i.e., X is reduced. □

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